INHOMOGENEOUS MULTIPARAMETER JORDANIAN QUANTUM GROUPS BY CONTRACTION

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Received XXX

It is known that the inhomogeneous quantum group $IGL_{q,r}(2)$ can be constructed as a quotient of the multiparameter q–deformation of $GL(3)$. We show that a similar result holds for the inhomogeneous Jordanian deformation and exhibit its Hopf structure.

1 Introduction

It is well–known [1] that, analogous to the classical group–theoretical method, the q–deformation of $IGL(2)$ can be constructed by factoring out a certain two–sided Hopf ideal from the multiparameter q–deformation of $GL(3)$. This is an interesting procedure, allowing, for example, the construction of a differential calculus on the quantum plane by a reduction of the differential calculus on the quantum group. In this paper, we apply the same construction to the Jordanian deformation. The multiparameter Jordanian deformation of $GL(3)$ is first produced by a contraction from the corresponding q–deformation and this is then used to construct the inhomogeneous group by factorisation. The Hopf -structure of $IGL_J(2)$ is given explicitly and we show that it is possible to derive from this a coaction of a modified version of $GL_J(2)$ on the Jordanian quantum plane.

Note: In this paper, we denote q–deformed structures using the (multiparameter) subscript Q and structures that have been contracted to the Jordanian form are written with a subscript J (e.g $GL_Q(3)$ and $GL_J(3)$).

2 The R–matrix for $\mathbf{GL}_Q\mathbf{(2)}$

Following Aschieri and Castellani [1], the R–matrix for $GL_Q(3)$ (where $Q = \{r, s, p, q\}$) can be written as

$$R_Q(3) = \begin{pmatrix} r & & & \\ & S^{-1} & & \\ & \Lambda & S & \\ & & & R_Q(2) \end{pmatrix} \tag{1}$$

where $S = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, $\Lambda = \begin{pmatrix} r - r^{-1} & 0 \\ 0 & r - r^{-1} \end{pmatrix}$ and

$$R_Q(2) = \begin{pmatrix} r & & & \\ & s & & \\ & r - r^{-1} & s^{-1} & \\ & & & r \end{pmatrix} \tag{2}$$

The matrix indices of $R_Q(3)$ run, in order, through the set (11), (12), (13), (21), (31), (22), (23), (32), (33). This numbering system is chosen to clearly show the embedding of the $R_Q(2)$ matrix in the $R_Q(3)$ matrix which, in turn, allows the Hopf structure of larger algebra to be analysed in terms of the simpler one. The Hopf structure of $GL_Q(3)$ is given by the RTT relations with T–matrix

$$\mathcal{T} = \begin{pmatrix} f & \theta & \phi \\ x & a & b \\ y & c & d \end{pmatrix} \tag{3}$$

and the multiparameter inhomogeneous q–deformation $IGL_Q(2)$ is the quantum homogeneous space

$$IGL_Q(2) = GL_Q(3)/H \tag{4}$$

where H is the two–sided Hopf ideal generated by the T–matrix elements $\{\theta, \phi\}$.

3 The Contraction Procedure

The R–matrix of the Jordanian (or h–deformation) can be viewed as a singular limit of a similarity transformation on the q–deformation R–matrix [2][3]. Let $g(\eta)$ be a matrix dependent on a contraction parameter η which is itself a function of one of the deformation parameters of the q–deformed algebra. This can be used to define a transformed q–deformed R–matrix

$$\tilde{R}_J = (g^{-1} \otimes g^{-1}) R_Q (g \otimes g) \tag{5}$$

The R–matrix of the Jordanian deformation is then obtained by taking a limiting value of the parameter η. Even though the contraction parameter η is undefined in this limit, the new R–matrix is finite and gives rise to a new quantum group structure through the RTT–relations. For example, in the contraction process which takes $GL_q(2)$ to $GL_h(2)$, the contraction matrix is

$$g(\eta) = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \tag{6}$$

where $\eta = \frac{h}{1-q}$ with h a new free parameter.

It has been shown by Alishahiha [3] that, in the extension of this procedure to the construction of $GL_J(3)$, there are essentially two choices of contraction matrix.

The first has been used in a number of papers, e.g. by Quesne [4] and takes the form

$$G' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \eta & 0 & 1 \end{pmatrix} \tag{7}$$

There is, however, a second choice (also mentioned in [3] but not pursued there since it gives trivial results for the single–parameter q–deformation)

$$G = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \tag{8}$$

where g is the 2×2 contraction matrix

$$g(\eta) = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \tag{9}$$

with $\eta = \frac{r}{1-q}$. In this present work, we take G as our contraction matrix because, unlike the matrix G', after contraction it allows a non–trivial embedding of $R_J(2)$ in $R_J(3)$ in a manner similar to the q–deformed case. It is then possible to perform the quotient construction for the inhomogeneous quantum group.

If the similarity transformation is made using the matrix G, we obtain

$$\begin{aligned} R_J(3) &= \lim_{r\to 1} \begin{pmatrix} r & & & \\ & g^{-1}S^{-1}g & & \\ & \Lambda & g^{-1}Sg & \\ & & & (g^{-1}\otimes g^{-1})R_Q(g\otimes g) \end{pmatrix} & (10) \\ &= \begin{pmatrix} 1 & & & \\ & K^{-1} & & \\ & & K & \\ & & & R_J(2) \end{pmatrix} & (11) \end{aligned}$$

where K is the matrix $\begin{pmatrix} p & 0 \\ k & p \end{pmatrix}$ and $R_J(2)$ is the R–matrix for the multiparameter Jordanian deformation of $GL(2)$

$$\begin{pmatrix} 1 & & & \\ m & 1 & & \\ -m & 0 & 1 & \\ mn & n & -n & 1 \end{pmatrix} \tag{12}$$

The free parameters $\{m, n, k\}$ appear as limits in the contraction process while the parameter $\{p\}$ survives the contraction process. The result is a four parameter Jordanian defomation of $GL(3)$.

4 Multiparameter Jordanian Deformation of GL(3)

We denote the T–matrix for the Jordanian deformation by

$$\mathcal{T} = \begin{pmatrix} f & \theta & \phi \\ x & a & b \\ y & c & d \end{pmatrix} = \begin{pmatrix} f & \Theta \\ X & T \end{pmatrix} \tag{13}$$

where $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, and $\Theta = \begin{pmatrix} \theta, \phi \end{pmatrix}$.

4.1 Algebra Relations

The algebra structure of the quantum group is obtained through the RTT–procedure. For the commutation relations between the elements of the matrix T, we have the usual relations between the generators of the multiparametric Jordanian deformation of $GL(2)$:

$$\begin{array}{lll} [a,b] = nb^2 & [a,c] = m(\delta - a^2) & [a,d] = nbd - mba \\ [b,d] = -mb^2 & [b,c] = -mba - ndb & [c,d] = n(d^2 - \delta) \end{array} \tag{14}$$

where δ is the quantum determinant of the submatrix T

$$\delta = ad - bc - nbd \tag{15}$$

with commutation relation

$$[\delta, a] = (m-n)\delta b \quad [\delta, b] = 0 \quad [\delta, c] = (m-n)(\delta d - a\delta) \quad [\delta, d] = (n-m)\delta b \tag{16}$$

Thus δ is not central in the (sub–Hopf) algebra generated by elements of T unless $m = n$.

The relations between T and f are given by

$$[a,f] = \tfrac{k}{p} fb \quad [b,f] = 0 \quad [c,f] = \tfrac{k}{p}(fd - af) \quad [d,f] = -\tfrac{k}{p} bf \tag{17}$$

those between T and X are

$$\begin{array}{ll} [a,x]_p = kxb & [b,x]_p = 0 \\ [c,x]_p = kxd + max & [d,x]_p = mbx \\ [a,y]_p = kyb - max & [b,y]_p = -mbx \\ [c,y]_p = kyd + ncx - nay - mnax & [d,y]_p = ndx - nby - mnbx \\ \delta x = p^2 x\delta & \delta y = p^2 y\delta + (n-m)\delta x \end{array} \tag{18}$$

while those between f and X give

$$[f,x]_p = 0 \quad [f,y]_p = -kxf \tag{19}$$

The commutation relations between the elements of X are the usual relations for the Jordanian quantum plane $C_J(2)$:

$$[x, y] = -mx^2 \tag{20}$$

There are also similar commutation relations between the elements of T, f and Θ, as well as cross-relations between X and Θ.

4.2 Coalgebra Relations and Antipode

The coalgebraic structure of the Hopf algebra is the usual one:

$$\Delta(\mathcal{T}) = \mathcal{T}\dot{\otimes}\mathcal{T} \qquad \epsilon(\mathcal{T}) = I_3 \tag{21}$$

with antipode

$$S(\mathcal{T}) = \begin{pmatrix} e & -e\Theta T^{-1} \\ -T^{-1}Xe & T^{-1}Xe\Theta T^{-1} + T^{-1} \end{pmatrix} \tag{22}$$

where we append to the algebra, the element $e = (f - \Theta T^{-1}X)^{-1}$. In terms of these elements, the quantum determinant of the T-matrix $\mathcal{T}$ is

$$\mathcal{D} = \det(\mathcal{T}) = e^{-1}\delta \tag{23}$$

and so, in the usual way, we can add $\xi = \mathcal{D}^{-1}$ to the algebra to obtain the full Hopf algebra.

5 The Inhomogeneous Multiparameter Jordanian Quantum Group $\mathbf{IGL}_J\mathbf{(2)}$

We define H to be the space of all monomials containing at least one element of Θ. It is straightforward to prove the following :

1. H is a two–sided ideal in $GL_J(3)$.
2. H is a co–ideal i.e. $\Delta(H) \subseteq H \otimes GL_J(3) + GL_J(3) \otimes H$ and $\epsilon(H) = 0$.
3. $S(H) \subseteq H$.

Thus H is a two–sided Hopf ideal and so we can define a canonical projection from $GL_J(3)$ to the quotient space $GL_J(3)/H$ which respects the Hopf–algebraic structure (i.e. the RTT–relations). Consequently the quotient is a Hopf algebra which we denote $IGL_J(2)$.

The algebra sector for this quantum group has commutation relations formally obtained from $GL_J(3)$ by setting the generator set $\Theta = 0$ and this gives rise to the commutation relations explicitly detailed in the previous section. The T–matrix for the coalgebra is given by

$$\mathcal{T} = \begin{pmatrix} f & 0 \\ X & T \end{pmatrix} \tag{24}$$

which gives the coproduct

$$\Delta(\mathcal{T}) = \mathcal{T}\dot{\otimes}\mathcal{T} = \begin{pmatrix} f \otimes f & 0 \\ T\dot{\otimes}X + X\dot{\otimes}f & T\dot{\otimes}T \end{pmatrix} \tag{25}$$

counit $\epsilon(\mathcal{T}) = I_3$ and antipode

$$S(\mathcal{T}) = \begin{pmatrix} f^{-1} & 0 \\ -T^{-1}Xf^{-1} & T^{-1} \end{pmatrix} \tag{26}$$

The quantum determinant $\mathcal{D} = f\delta$ is group–like but, since f is not central, it cannot be made simultaneously central with δ unless the whole algebraic structure collapses to a trivial extension of the single–parameter Jordanian deformation of $GL(2)$. This is analagous to the situation in the q–deformed case.

This procedure also shows that it is possible to view the Jordanian quantum plane, $C_J(2)$, as the quantum homogeneous space $IGL_J(2)/GL_J(2)^*$ where $GL_J(2)^*$ is the Hopf algebra formed by appropriately appending the "dilatation element" f to $GL_J(2)$. The comultiplication in $IGL_J(2)$ can then be viewed as a coaction of the quantum group $GL_J(2)^*$ on the quantum plane $C_J(2)$ generated by the elements X. However, unlike the usual case, there is a non–trivial braiding between the elements of the quantum group and quantum plane.

6 Conclusion

We have shown that it is possible to construct the inhomogeneous Jordanian deformation $IGL_J(2)$ as a quotient group by factoring out a Hopf ideal from $GL_J(3)$. It would be of interest to construct the differential calculus on the Jordanian quantum plane by a reduction of the bicovariant differential caculus on $GL_J(3)$. This would be allow the investigation of physical models with $GL_J(N)$ symmetry similar to that of Cho *et al* [5] and Madore and Steinacker [6]. Work on this problem is underway.

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